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Free Vibration of Pretwisted, Cantilevered Composite Shallow Conical Shells

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The free vibration analysis of composite shallow conical shells, incorporating the effects of pretwist, is presented. The analysis focuses on shells made of symmetrically and unsymmetrically laminated E-glass/epoxy with four- and eight-ply laminates. This method is valid, however, for any other composite with an arbitrary number of layers and fiber orientation. An extremum energy principle is employed to derive the eigenvalue equation, and a flexible, global admissible function is developed to account for the geometric boundary conditions. New vibration frequency parameters and mode shapes are presented.

I. Introduction

PRETWISTED blades are key structural components in power-generating turbines in the aeronautical and aerospace industries. There are many factors affecting the dynamic behavior of these blades, such as material properties and angle of pretwist. The effect of pretwist on the vibration characteristics of these turbomachinery blades has long been a subject of interest to design engineers, as has the use of composite materials to achieve higher strength, greater durability, and less weight. The use of composites in designing turbomachinery blades has great potential because there are virtually unlimited ways of tailoring the mechanical properties of composites to suit design requirements.

The vibration of turbomachinery blades has received much attention for decades as reviewed by Rao[1–5] and Leissa.[6,7] The conventional models for turbomachinery blades are beams,[8,9] plates,[10–12] and shallow shells.[13–19] The beam model is accurate for slender blades. For blades with a small aspect ratio, the plate and shell models are more appropriate. Shell models are preferable to plate models because the effect of surface curvature is considered. The shallow cylindrical shell model has been applied by Leissa and his associates to study the vibration of blades with uniform[13,14] or variable[15] thickness. A similar analysis also has been undertaken by Liew and Lim[16] for cylindrical shells with generally varying thickness. One major deficiency of the cylindrical shallow shell model, however, is the constant chordwise curvature. A better model of an actual turbomachinery blade should feature shallow shell with not only nonuniform planform but also variable chordwise curvature. Thus an open conical shell model is more appropriate. The vibration of open conical shells has been reported respectively for untwisted shells with uniform thickness,[17] pretwisted shells with uniform thickness,[18] and pretwisted shells with variable thickness.[19] Only isotropic shells are considered in Refs. 13–19.

The design of turbomachinery blades is seldom confined to isotropic materials. Very frequently composite materials are preferred because of advantages in strength, durability, and weight. An in-depth knowledge of the effects of layer lamination and fiber orientation on the vibration of these structures is therefore necessary.

The study of vibration of composite shells is extensive and has been reviewed by Kapania,[20] Mirza,[21] and Qatu.[22] However, to the authors' knowledge, the vibration of composite turbomachinery blades using a pretwisted shallow conical shell model has not been investigated. This papers addresses this subject, investigating the effects of the angle of pretwist, angle of fiber orientation, and symmetric and unsymmetric lamination on the vibration characteristics of cantilevered, open shallow conical shells.

The Ritz extremum energy principle is employed to formulate the governing eigenvalue equation, and the kinematically oriented *pb*-2 admissible functions, previously used in plate and shell analysis, are further extended to composite shells with multiple plies. Previously unavailable results for different laminations are presented for design and comparison purposes.

II. Energy Functional and Governing Eigenvalue Equation

A. Geometry of a Conical Shell

Consider a thin, untwisted shallow conical shell with midsurface length a, reference width b_o, thickness h, cone length s, vertex angle θ_v, and base subtended angle θ_o, as illustrated in Fig. 1. The cone base can be assumed to be elliptical with minor and major radii α_o and β_o because the shell is shallow. The radius of curvature in the chordwise direction $R_y(x, y)$ is a parameter varying in the x and y directions. The variation of this curvature in the x direction is linear. There is no curvature along the spanwise direction ($R_x = \infty$). For a shallow conical shell with pretwist as shown in Fig. 2, the radius and angle of twist can be denoted as R_{xy} and ψ, respectively. This cantilevered shell is clamped along $x = 0$.

The midsurface geometry of a pretwisted shallow conical shell is rather complex. Considering an untwisted shallow conical shell as shown in Fig. 1, the reference major radius β_o and the major and minor radii β and α at any vertical cross section are

$$\beta_o = s\tan(\theta_v/2) \tag{1a}$$

$$\beta = \tan(\theta_v/2)(s - x) \tag{1b}$$

$$\alpha = \frac{b\beta\tan(\theta_o/2)}{\sqrt{4\beta^2\tan^2(\theta_o/2) - b^2}} \tag{1c}$$

where

$$b = b_o[1 - (x/s)] \tag{2a}$$

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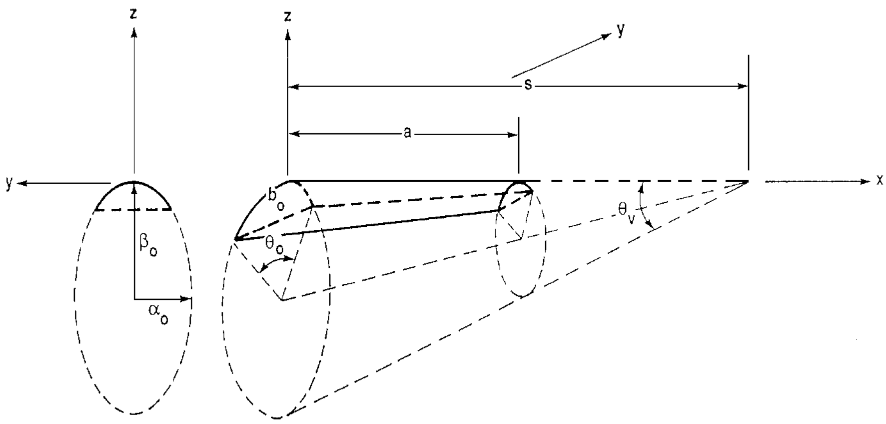

Fig. 1 Geometry of an untwisted laminated CFFF shallow conical shell.

and

$$b_o = 2s\sin\frac{\theta_v}{2}\sqrt{\frac{\tan^2\theta_o/2}{\cos^2\theta_v/2+\tan^2\theta_o/2}} \tag{2b}$$

The equation of an ellipse at any perpendicular cross section is

$$(y/\alpha)^2 + [(z+\beta)/\beta]^2 = 1 \tag{3}$$

Taking the first and second derivatives of z with respect to y and making use of the definition of curvature

$$k = \frac{\mathrm{d}^2 z}{\mathrm{d}y^2}\Bigg/\left[1+\left(\frac{\mathrm{d}z}{\mathrm{d}y}\right)^2\right]^{\frac{3}{2}}$$

the chordwise radius of curvature is

$$R_y(x,y) = \left|\frac{1}{k}\right| = \alpha^2\beta^2\left[\frac{1}{\beta^2}+\frac{y^2}{\alpha^2}\left(\frac{1}{\alpha^2}-\frac{1}{\beta^2}\right)\right]^{\frac{3}{2}} \tag{4}$$

In the presence of small angle of twist ψ, a conical shell as shown in Fig. 2 exhibits an additional surface curvature represented by the radius of twist R_{xy}, which is related to the angle of twist by

$$R_{xy} = -a/\tan\psi \tag{5}$$

B. Energy Functional and Governing Eigenvalue Equation

The total strain energy $\mathcal{U}$ of a laminated conical shell can be expressed as

$$\mathcal{U} = \mathcal{U}_{or} + \mathcal{U}_{es} + \mathcal{U}_{bs} + \mathcal{U}_{bt} \tag{6}$$

where the strain energy of uncoupled orthotropic characteristics of the material (i.e., A_{11}, A_{12}, A_{22}, A_{66}, D_{11}, D_{12}, D_{22}, and D_{66}) is

$$\begin{aligned}\mathcal{U}_{or} = \frac{1}{2}\iint_A \Bigg[& A_{11}\left(\frac{\partial U}{\partial x}\right)^2 + A_{22}\left(\frac{\partial V}{\partial y}\right)^2 + A_{66}\left(\frac{\partial U}{\partial y}\right)^2 \\ & + A_{66}\left(\frac{\partial V}{\partial x}\right)^2 + 2A_{12}\left(\frac{\partial U}{\partial x}\frac{\partial V}{\partial y}\right) + \frac{2A_{12}}{R_y(x,y)}\frac{\partial U}{\partial x}W \\ & + \frac{2A_{22}}{R_y(x,y)}\frac{\partial V}{\partial y}W + \frac{A_{22}}{R_y^2(x,y)}W^2 + 2A_{66}\left(\frac{\partial U}{\partial y}\frac{\partial V}{\partial x}\right) \\ & + \frac{4A_{66}}{R_{xy}}\frac{\partial U}{\partial y}W + \frac{4A_{66}}{R_{xy}}\frac{\partial V}{\partial y}W + \frac{4A_{66}}{R_{xy}^2}W^2 + D_{11}\left(\frac{\partial^2 W}{\partial x^2}\right)^2 \\ & + 2D_{12}\left(\frac{\partial^2 W}{\partial x^2}\right)\left(\frac{\partial^2 W}{\partial y^2}\right) + D_{22}\left(\frac{\partial^2 W}{\partial y^2}\right)^2 \\ & + 4D_{66}\left(\frac{\partial^2 W}{\partial x\partial y}\right)^2\Bigg]\,\mathrm{d}x\,\mathrm{d}y\end{aligned} \tag{7a}$$

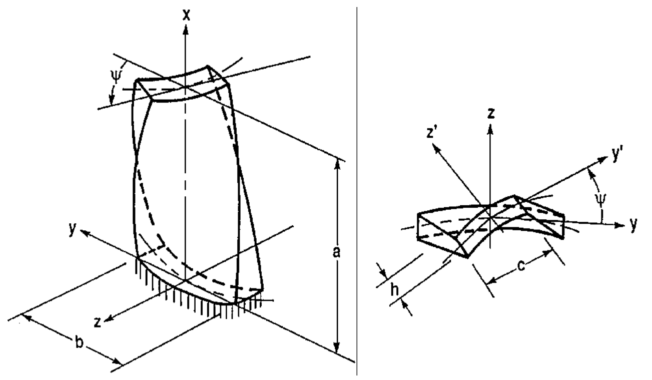

Fig. 2 Geometry of a pretwisted laminated CFFF shallow conical shell.

strain energy of extension–shearing coupling (i.e., A_{16} and A_{26}) is

$$\begin{aligned}\mathcal{U}_{es} = \iint_A \Bigg[& A_{16}\frac{\partial U}{\partial x}\frac{\partial U}{\partial y} + A_{16}\frac{\partial U}{\partial x}\frac{\partial V}{\partial x} + \frac{2A_{16}}{R_{xy}}\frac{\partial U}{\partial x}W \\ & + A_{26}\frac{\partial U}{\partial y}\frac{\partial V}{\partial y} + A_{26}\frac{\partial V}{\partial x}\frac{\partial V}{\partial y} + \frac{A_{26}}{R_y(x,y)}\frac{\partial U}{\partial y}W \\ & + \frac{A_{26}}{R_y(x,y)}\frac{\partial V}{\partial x}W + \frac{2A_{26}}{R_{xy}}\frac{\partial V}{\partial y}W + \frac{2A_{26}}{R_y(x,y)R_{xy}}W^2\Bigg]\,\mathrm{d}x\,\mathrm{d}y\end{aligned} \tag{7b}$$

strain energy of bending–stretching coupling (i.e., B_{ij}, $i, j = 1, \ldots, 6$) is

$$\begin{aligned}\mathcal{U}_{bs} = -\iint_A \Bigg[& B_{11}\frac{\partial U}{\partial x}\frac{\partial^2 W}{\partial x^2} + B_{12}\frac{\partial U}{\partial x}\frac{\partial^2 W}{\partial y^2} + B_{16}\frac{\partial U}{\partial y}\frac{\partial^2 W}{\partial x^2} \\ & + 2B_{16}\frac{\partial U}{\partial x}\frac{\partial^2 W}{\partial x\partial y} + B_{26}\frac{\partial U}{\partial y}\frac{\partial^2 W}{\partial y^2} + 2B_{66}\frac{\partial U}{\partial y}\frac{\partial^2 W}{\partial x\partial y} \\ & + B_{12}\frac{\partial V}{\partial y}\frac{\partial^2 W}{\partial x^2} + B_{22}\frac{\partial V}{\partial y}\frac{\partial^2 W}{\partial y^2} + B_{16}\frac{\partial V}{\partial x}\frac{\partial^2 W}{\partial x^2} \\ & + 2B_{26}\frac{\partial V}{\partial y}\frac{\partial^2 W}{\partial x\partial y} + B_{26}\frac{\partial V}{\partial x}\frac{\partial^2 W}{\partial y^2} + 2B_{66}\frac{\partial V}{\partial x}\frac{\partial^2 W}{\partial x\partial y} \\ & + \frac{B_{12}}{R_y(x,y)}W\frac{\partial^2 W}{\partial x^2} + \frac{2B_{16}}{R_{xy}}W\frac{\partial^2 W}{\partial x^2} + \frac{B_{22}}{R_y(x,y)}W\frac{\partial^2 W}{\partial y^2} \\ & + \frac{2B_{26}}{R_{xy}}W\frac{\partial^2 W}{\partial y^2} + \frac{2B_{26}}{R_y(x,y)}W\frac{\partial^2 W}{\partial x\partial y} + \frac{4B_{66}}{R_{xy}}W\frac{\partial^2 W}{\partial x\partial y}\Bigg]\,\mathrm{d}x\,\mathrm{d}y\end{aligned} \tag{7c}$$

and strain energy of bending–twisting coupling (i.e., D_{16} and D_{26}) is

$$\mathcal{U}_{bt} = 2\iint_A \left[D_{16}\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial x \partial y} + D_{26}\frac{\partial^2 W}{\partial y^2}\frac{\partial^2 W}{\partial x \partial y} \right] \mathrm{d}x\,\mathrm{d}y \tag{7d}$$

where the integration is on the planform area A of the shallow conical shell. The laminate stiffness coefficients A_{ij}, B_{ij}, and D_{ij} for the k-ply[23] are

$$A_{ij} = \sum_{k=1}^{n} (\bar{Q}_{ij})_k (h_k - h_{k-1}) \qquad (i, j = 1, 2, 6) \tag{8a}$$

$$B_{ij} = \frac{1}{2}\sum_{k=1}^{n} (\bar{Q}_{ij})_k (h_k^2 - h_{k-1}^2) \qquad (i, j = 1, 2, 6) \tag{8b}$$

$$D_{ij} = \frac{1}{3}\sum_{k=1}^{n} (\bar{Q}_{ij})_k (h_k^3 - h_{k-1}^3) \qquad (i, j = 1, 2, 6) \tag{8c}$$

in which $\bar{Q}_{ij}$ $(i, j = 1, 2, 6)$ is the transformed stiffness depending on the ply stiffness constant Q_{ij} $(i, j = 1, 2, 6)$ and fiber orientation angle. The bending–stretching coupling vanishes ($B_{ij} = 0$) for symmetrically laminated shells.

The kinetic energy for free vibration is given by

$$\mathcal{T} = \frac{\rho h \omega^2}{2}\iint_A (U^2 + V^2 + W^2)\,\mathrm{d}x\,\mathrm{d}y \tag{9}$$

where ρ is the mass density per unit volume and ω is the rotational frequency.

The inplane and transverse displacement amplitude functions can be approximated by a series of orthogonally generated two-dimensional polynomials:

$$U(\xi, \eta) = \sum_{i=1}^{m} C_i^u \phi_i^u(\xi, \eta) \tag{10a}$$

$$V(\xi, \eta) = \sum_{i=1}^{m} C_i^v \phi_i^v(\xi, \eta) \tag{10b}$$

$$W(\xi, \eta) = \sum_{i=1}^{m} C_i^w \phi_i^w(\xi, \eta) \tag{10c}$$

where C_i^u, C_i^v, and C_i^w are the unknown coefficients and ϕ_i^u, ϕ_i^v, and ϕ_i^w are the corresponding pb-2 admissible functions in terms of a nondimensional coordinate system defined as

$$\xi = x/a \tag{11a}$$

$$\eta = y/b_o \tag{11b}$$

in which a and b_o are the span and width of the shell as shown in Fig. 1.

The maximum strain energy $\mathcal{U}_{max}$ and maximum kinetic energy $\mathcal{T}_{max}$ in a vibratory cycle occur at maximum displacement and maximum velocity, respectively. Following the Ritz extremum energy principle, the energy functional

$$\mathcal{F} = \mathcal{U}_{max} - \mathcal{T}_{max} \tag{12}$$

is minimized with respect to the unknown coefficients to obtain the following eigenvalue equation:

$$(K - \Lambda^2 M)\{C\} = \{0\} \tag{13}$$

where K and M are the stiffness and mass matrices expressed as

$$K = \begin{bmatrix} k^{uu} & k^{uv} & k^{uw} \\ & k^{vv} & k^{vw} \\ \text{sym} & & k^{ww} \end{bmatrix} \tag{14a}$$

$$M = \begin{bmatrix} m^{uu} & [0] & [0] \\ & m^{vv} & [0] \\ \text{sym} & & m^{ww} \end{bmatrix} \tag{14b}$$

the vector of unknown coefficients is

$$\{C\} = \begin{Bmatrix} \{c^u\} \\ \{c^v\} \\ \{c^w\} \end{Bmatrix} \tag{14c}$$

and the nondimensional frequency parameter is

$$\Lambda = (b_o/h)\sqrt{12(1 - \nu_{12}\nu_{21})}\,\lambda \tag{15a}$$

$$\lambda = \omega a\sqrt{\rho/E_{11}} \tag{15b}$$

The pb-2 admissible functions introduced in Eqs. (10a–10c) consist of the product of terms of a two-dimensional orthogonally generated polynomial (p-2) and appropriate basic functions (b), i.e.,

$$\phi_i^\alpha(\xi, \eta) = f_i(\xi, \eta)\phi_b^\alpha - \sum_{j=1}^{i-1} \Xi_{ij}^\alpha \phi_j^\alpha \tag{16}$$

where

$$\Xi_{ij}^\alpha = \frac{{}_1\Delta_{ij}^\alpha}{{}_2\Delta_j^\alpha} \tag{17a}$$

$${}_1\Delta_{ij}^\alpha = \iint_A f_i(\xi, \eta)\phi_b^\alpha \phi_j^\alpha\,\mathrm{d}\xi\mathrm{d}\eta \tag{17b}$$

$${}_2\Delta_j^\alpha = \iint_A (\phi_j^\alpha)^2\,\mathrm{d}\xi\mathrm{d}\eta \tag{17c}$$

in which $\alpha = u$, v, or w and

$$f_i(\xi, \eta) = 1, \xi, \eta, \xi^2, \xi\eta, \eta^2, \xi^3, \xi^2\eta, \xi\eta^2, \eta^3, \ldots \tag{17d}$$

form a complete set of p-2 functions. Note that α here is only a dummy variable that is not related to the minor radius in Fig. 1. The basic functions are ϕ_b^α ($\alpha = u$, v, or w) defined as the products of the equations of the continuous piecewise boundary geometries of the shell planform, each of which is raised to an appropriate basic power that corresponds to its geometric boundary condition.

In this study, the basic functions for the cantilevered conical shallow shell are

$$\phi_b^u = \xi \tag{18a}$$

$$\phi_b^v = \xi \tag{18b}$$

$$\phi_b^w = \xi^2 \tag{18c}$$

which satisfy the geometric boundary conditions at the clamped edge $\xi = 0$.

III. Frequency Responses

This study focuses on cantilevered shallow conical shells, denoted as CFFF, clamped along $\xi = 0$ and free on the other edges. The shells are made of E-glass/epoxy (E/E). The mechanical properties of this material are

$$E_{11} = 60.7\,\text{GPa} \qquad E_{22} = 24.8\,\text{GPa}$$

$$G_{12} = 12.0\,\text{GPa} \qquad \nu_{12} = 0.23$$

where E_{11} and E_{22} are the Young's moduli parallel and perpendicular, respectively, to the fiber; G_{12} is the shear modulus; and ν_{12} is Poisson's ratio.

Although only cantilevered E/E shells are considered, this analysis is valid for other types of laminated composite material. Similarly, other classes of boundary conditions also can be incorporated into the formulation by substituting other appropriate basic functions for ϕ_b^u, ϕ_b^v, and ϕ_b^w in Eqs. (18a–18c). A detailed description of the choice of basic functions can be found in Lim and Liew.[17]

Table 1 Frequency parameters $\lambda = \omega a \sqrt{(\rho/E_{11})}$ for a CFFF E/E shallow conical shell ($a/h = 100.0$, $a/b_o = 1.5$, $\theta_v = 15$ deg, $\theta_o = 30$ deg, $\psi = 15$ deg, and $\theta_f = 30$ deg)

		Mode frequencies			
Lamination	p	1	2	3	4
$[(-\theta_f, \theta_f)_4]_{sym}$	10	0.022183	0.064474	0.12280	0.13857
	11	0.022179	0.064466	0.12278	0.13857
	12	0.022175	0.064460	0.12277	0.13856
	13	0.022173	0.064456	0.12276	0.13856
	14	0.022171	0.064453	0.12275	0.13856
	15	0.022170	0.064450	0.12275	0.13856
$[(-\theta_f, \theta_f)_4]_{unsym}$	10	0.021375	0.063898	0.12020	0.13728
	11	0.021370	0.063890	0.12019	0.13727
	12	0.021367	0.063884	0.12018	0.13727
	13	0.021365	0.063880	0.12017	0.13726
	14	0.021363	0.063877	0.12016	0.13726
	15	0.021362	0.063875	0.12016	0.13726
$[(\theta_f, -\theta_f)_4]_{sym}$	10	0.020609	0.062984	0.11768	0.13643
	11	0.020605	0.062976	0.11767	0.13643
	12	0.020603	0.062971	0.11766	0.13642
	13	0.020600	0.062967	0.11765	0.13642
	14	0.020599	0.062963	0.11765	0.13642
	15	0.020598	0.062961	0.11764	0.13641
$[(\theta_f, -\theta_f)_4]_{unsym}$	10	0.021429	0.063673	0.12077	0.13720
	11	0.021424	0.063666	0.12076	0.13720
	12	0.021421	0.063660	0.12074	0.13719
	13	0.021419	0.063656	0.12073	0.13719
	14	0.021417	0.063653	0.12073	0.13719
	15	0.021416	0.063651	0.12072	0.13718

A. Convergence Study

The convergence of eigenvalues for a pretwisted, cantilevered conical shell with eight-ply symmetric and unsymmetric laminations is presented in Table 1. The notation $[(-\theta_f, \theta_f)_4]_{sym}$ indicates an eight-ply symmetric laminate with fiber of the bottom laminae making a $-\theta_f$ angle with respect to the x axis. The eigenvalues corresponding to symmetric (or unsymmetric) laminations are not identical for the bottom laminae making $-\theta_f$ or θ_f with respect to the x axis. For example, the frequencies for $[(-\theta_f, \theta_f)_4]_{sym}$ and $[(\theta_f, -\theta_f)_4]_{sym}$ are 0.022170 and 0.020598 in Table 1. This is not paradoxical because the symmetry of positive and negative fiber angles does not exist for a shell with pretwist.

The degree p of the pb-2 admissible functions for u, v, and w has been increased from 10 to 15. The number of terms m for u, v, and w as expressed in Eq. (17d) can be determined by $m = (p + 1)(p + 2)/2$. The total number of terms in the pb-2 admissible functions in Table 1 increases from 66 to 136 for each of u, v, and w; and the determinant size of the eigenvalue Eq. (13) increases from $(66 \times 3) \times (66 \times 3)$ to $(136 \times 3) \times (136 \times 3)$.

It can be seen in Table 1 that the convergence of eigenvalues is excellent and the eigenvalues converge downward as p increases. It has been shown in many studies that the Ritz method produces downward-converging eigenvalues, indicating the existence of upperbound values.[11,12,16–19] Increasing the number of terms in the admissible functions results in shape functions with higher flexibility. Table 1 shows that $p = 15$ for u, v, and w produces excellent converged eigenvalues. Unless otherwise stated, all subsequent numerical results are computed using $p = 15$ or $m = 136$.

B. Frequency Parameters and Mode Shapes

The effects of fiber angle θ_f and angle of pretwist ψ on the fundamental frequency parameter λ are shown in Table 2 and Figs. 3–6 for eight-ply symmetrically and unsymmetrically laminated E/E. In Table 2, new results for eight-ply symmetric and unsymmetric laminates with $\psi = 15$ deg and θ_f from 0 to 90 deg are presented. As can be observed, the effect of symmetric or unsymmetric laminations on the fundamental λ is not very significant. The frequencies for symmetric and unsymmetric laminations are identical for $\theta_f = 0$ deg and $\theta_f = 90$ deg regardless of the number of plies because all of the fibers are aligned. In these cases, a multiple-ply laminate is equivalent to a single-ply laminate.

In Figs. 3–6, an increase in ψ results in decreasing fundamental λ for $\theta_f = 0, 30, 60,$ and 90 deg. This phenomenon has been observed in many investigations experimentally[9,10] and numerically[11,12,15] for plates. To further verify the results, finite element solutions have been obtained using LUSAS as shown in Fig. 3. The pretwisted conical shell is modeled using a total of 20×10 eight-node semiloof elements. Convergence of the finite element solutions has been checked. Agreement of finite element method and the present Ritz solutions is excellent, especially for $\psi = 15$ deg. For $\psi = 0$ deg, the agreement is generally good. The largest difference is for $\psi = 0$ deg and $\theta_f = 60$ deg, which is about 6% and is within acceptable

Table 2 Frequency parameters $\lambda = \omega a \sqrt{(\rho/E_{11})}$ for a CFFF E/E shallow conical shell ($a/h = 100.0$, $a/b_o = 1.5$, $\theta_v = 15$ deg, $\theta_o = 30$ deg, and $\psi = 15$ deg)

		Mode frequencies			
Lamination	θ_f	1	2	3	4
$[(-\theta_f, \theta_f)_4]_{sym}$	0	0.021271	0.066337	0.12163	0.13323
	15	0.022062	0.066762	0.12466	0.13359
	30	0.022170	0.064450	0.12275	0.13856
	45	0.020996	0.059658	0.11491	0.14857
	60	0.019264	0.055021	0.10636	0.15282
	75	0.017700	0.051586	0.099348	0.14479
	90	0.016933	0.050176	0.095991	0.14031
$[(-\theta_f, \theta_f)_4]_{unsym}$	0	0.021271	0.066337	0.12163	0.13323
	15	0.021431	0.066187	0.12220	0.13351
	30	0.021362	0.063875	0.12016	0.13726
	45	0.020405	0.059428	0.11332	0.14583
	60	0.018954	0.055037	0.10550	0.15022
	75	0.017578	0.051632	0.098925	0.14407
	90	0.016933	0.050176	0.095991	0.14031
$[(\theta_f, -\theta_f)_4]_{sym}$	0	0.021271	0.066337	0.12163	0.13323
	15	0.020872	0.065316	0.12032	0.13352
	30	0.020598	0.062961	0.11764	0.13641
	45	0.019763	0.058906	0.11113	0.14364
	60	0.018549	0.054819	0.10392	0.14780
	75	0.017396	0.051588	0.098095	0.14342
	90	0.016933	0.050176	0.095991	0.14031
$[(\theta_f, -\theta_f)_4]_{unsym}$	0	0.021271	0.066337	0.12163	0.13323
	15	0.021498	0.065920	0.12278	0.13355
	30	0.021416	0.063651	0.12072	0.13718
	45	0.020405	0.059361	0.11332	0.14588
	60	0.018923	0.055049	0.10521	0.15048
	75	0.017550	0.051648	0.098669	0.14434
	90	0.016933	0.050176	0.095991	0.14031

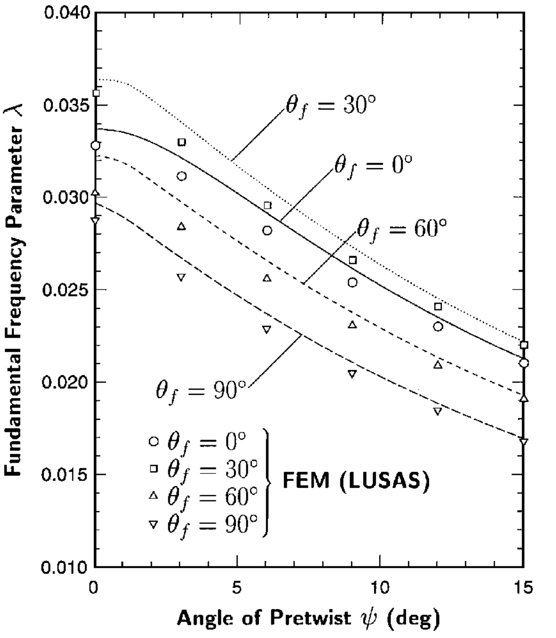

Fig. 3 Effects of pretwist on the fundamental frequency parameter for a CFFF eight-ply symmetrically laminated E/E shallow conical shell {$a/h = 100.0$, $a/b_o = 1.5$, $\theta_v = 15$ deg, $\theta_o = 30$ deg, and stacking sequence $[(-\theta_f, \theta_f)_4]_{sym}$}.

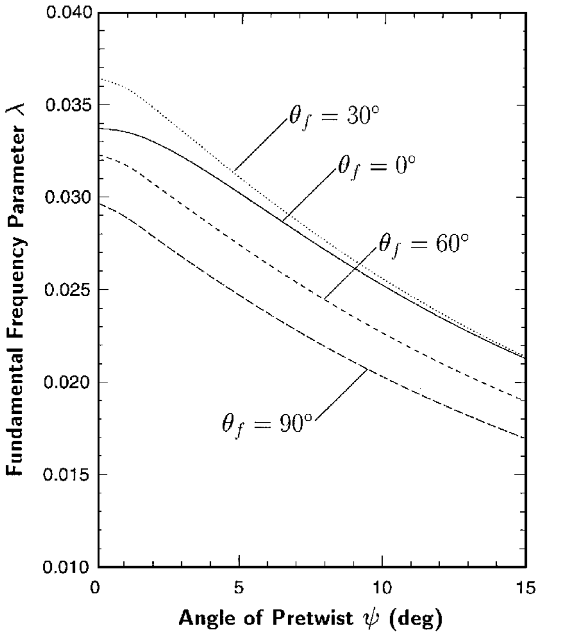

Fig. 4 Effects of pretwist on the fundamental frequency parameter for a CFFF eight-ply unsymmetrically laminated E/E shallow conical shell {$a/h = 100.0$, $a/b_o = 1.5$, $\theta_v = 15$ deg, $\theta_o = 30$ deg, and stacking sequence $[(-\theta_f, \theta_f)_4]_{\text{unsym}}$}.

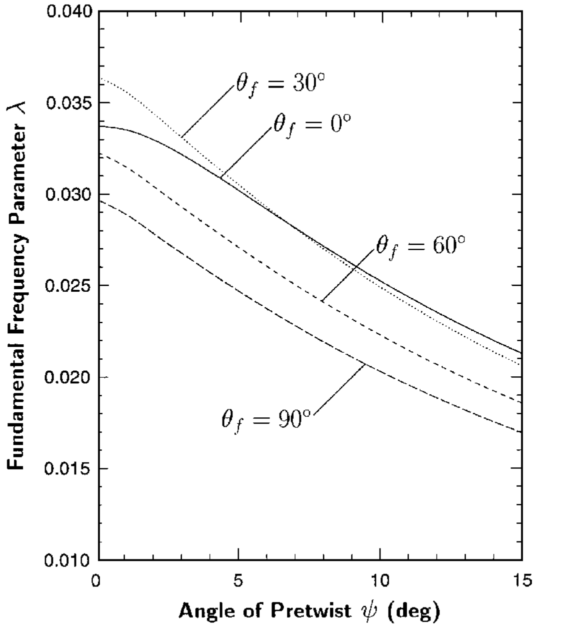

Fig. 5 Effects of pretwist on the fundamental frequency parameter for a CFFF eight-ply symmetrically laminated E/E shallow conical shell {$a/h = 100.0$, $a/b_o = 1.5$, $\theta_v = 15$ deg, $\theta_o = 30$ deg, and stacking sequence $[(\theta_f, -\theta_f)_4]_{\text{sym}}$}.

practical tolerance. It is also observed that for $\psi = 0$ deg (an untwisted shell), the highest fundamental λ corresponds to $\theta_f = 30$ deg and the lowest corresponds to $\theta_f = 90$ deg for the range of fiber angle investigated.

To enhance the physical aspects of vibration, a set of previously unavailable mode shapes is illustrated in contour and midsurface displacements in Fig. 7. These mode shapes correspond to a four-ply CFFF conical shell with $[(-\theta_f, \theta_f)_2]_{\text{sym}}$ lamination. The effects of pretwisted angle on the vibration mode shapes are presented in these illustrations. The shaded and unshaded regions in the contour plots refer to regions with positive and negative vibration displacement amplitude, respectively. The lines of demarcation are the nodal lines with zero vibration amplitude. For $\psi = 0$ deg, M-1 is a twisting

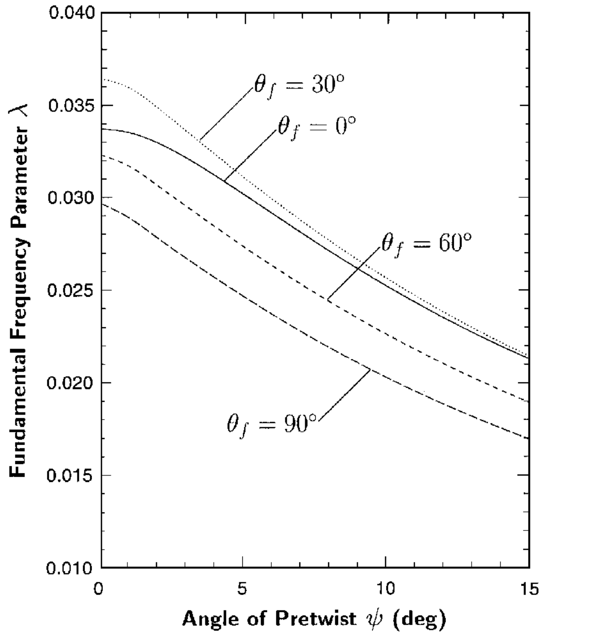

Fig. 6 Effects of pretwist on the fundamental frequency parameter for a CFFF eight-ply unsymmetrically laminated E/E shallow conical shell {$a/h = 100.0$, $a/b_o = 1.5$, $\theta_v = 15$ deg, $\theta_o = 30$ deg, and stacking sequence $[(\theta_f, -\theta_f)_4]_{\text{unsym}}$}.

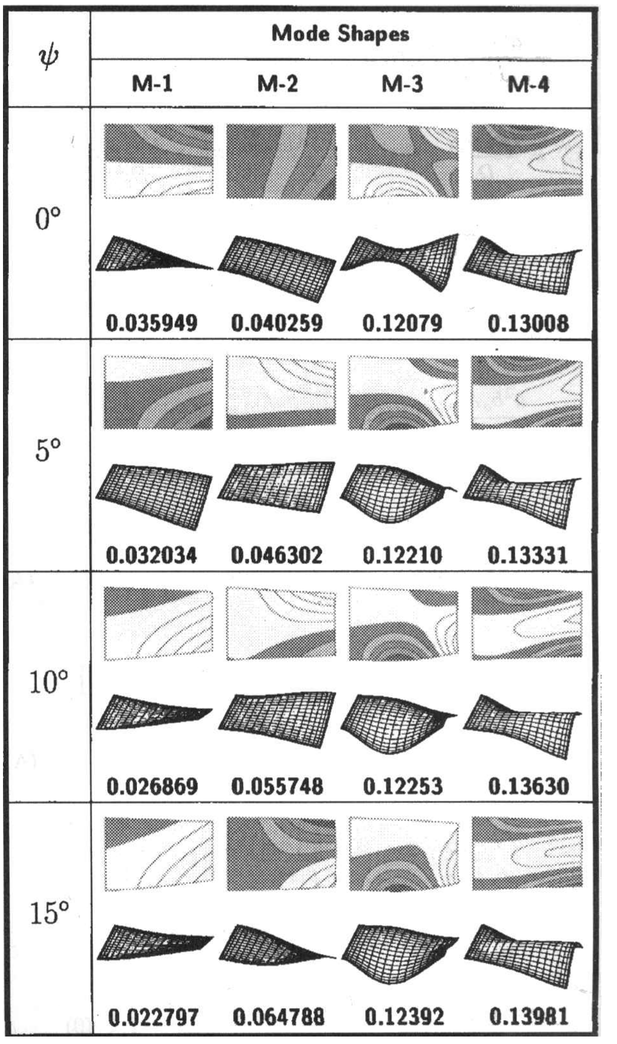

Fig. 7 Contour and midsurface displacement mode shapes for a CFFF four-ply E/E shallow conical shell {$a/h = 100.0$, $a/b_o = 1.5$, $\theta_v = 15$ deg, $\theta_o = 30$ deg, $\theta_f = 30$ deg, and stacking sequence $[(-\theta_f, \theta_f)_2]_{\text{sym}}$}.

mode, M-2 is a bending mode, and M-3 and M-4 are coupled bending and twisting modes. For pretwisted shells, there is no pure bending and twisting mode because all of the modes are strongly coupled.

IV. Conclusions

This study presents a method of modeling laminated, pretwisted shallow conical shells. It investigates the effects of pretwist and fiber orientation on free vibration, employing a computational Ritz extremum energy approach with a nondiscretized global element. Integral expressions for the strain and kinetic energies, incorporating the effects of chordwise-varying surface curvature, are formulated. Although the study focuses on E/E with four-ply and eight-ply lamination, the formulation is general and can be extended easily to any composite with an arbitrary number of plies and fiber orientation. Globally flexible *pb*-2 admissible functions satisfying the geometric boundary conditions at the outset are introduced.

New frequency parameters and mode shapes are presented. The convergence study shows downward-converging eigenvalues. The study shows that the frequencies are not identical, because of the presence of pretwist, for symmetrically (or unsymmetrically) laminated shells with the bottom plies making negative or positive angles with respect to the x axis. An increase in the angle of pretwist results in a lower fundamental frequency parameter. The vibration modes for pretwisted shells are strongly coupled.

Appendix: Elements in Stiffness and Mass Matrices

The elements in the stiffness and mass matrices are

$$k^{uu}_{ij} = \frac{b_o^2 A_{11}}{D_o} I^{1010}_{uiuj}(0) + \frac{ab_o A_{16}}{D_o}\left[I^{0110}_{uiuj}(0) + I^{1001}_{uiuj}(0)\right] + \frac{a^2 A_{66}}{D_o} I^{0101}_{uiuj}(0) \tag{A1a}$$

$$k^{uv}_{ij} = \frac{ab_o A_{12}}{D_o} I^{1001}_{uivj}(0) + \frac{b_o^2 A_{16}}{D_o} I^{1010}_{uivj}(0) + \frac{a^2 A_{26}}{D_o} I^{0101}_{uivj}(0) + \frac{ab_o A_{66}}{D_o} I^{0110}_{uivj}(0) \tag{A1b}$$

$$\begin{aligned} k^{uw}_{ij} = {} & \frac{ab_o^2 A_{12}}{\beta_o D_o} I^{1000}_{uiwj}(1) + \frac{2ab_o^2 A_{16}}{R_{xy} D_o} I^{1000}_{uiwj}(0) \\ & + \frac{a^2 b_o A_{26}}{\beta_o D_o} I^{0100}_{uiwj}(1) + \frac{2a^2 b_o A_{66}}{R_{xy} D_o} I^{0100}_{uiwj}(0) - \frac{b_o^2 B_{11}}{a D_o} I^{1020}_{uiwj}(0) \\ & - \frac{a B_{12}}{D_o} I^{1002}_{uiwj}(0) - \frac{b_o B_{16}}{D_o}\left[I^{0120}_{uiwj}(0) + 2 I^{1011}_{uiwj}(0)\right] \\ & - \frac{a^2 B_{26}}{b_o D_o} I^{0102}_{uiwj}(0) - \frac{2a B_{66}}{D_o} I^{0111}_{uiwj}(0) \end{aligned} \tag{A1c}$$

$$k^{vv}_{ij} = \frac{a^2 A_{22}}{D_o} I^{0101}_{vivj}(0) + \frac{ab_o A_{26}}{D_o}\left[I^{0110}_{vivj}(0) + I^{1001}_{vivj}(0)\right] + \frac{b_o^2 A_{66}}{D_o} I^{1010}_{vivj}(0) \tag{A1d}$$

$$\begin{aligned} k^{vw}_{ij} = {} & \frac{a^2 b_o A_{22}}{\beta_o D_o} I^{0100}_{viwj}(1) + \frac{ab_o^2 A_{26}}{\beta_o D_o} I^{1000}_{viwj}(1) \\ & + \frac{2a^2 b_o A_{26}}{R_{xy} D_o} I^{0100}_{viwj}(0) + \frac{2ab_o^2 A_{66}}{R_{xy} D_o} I^{1000}_{viwj}(0) \\ & - \frac{b_o B_{12}}{D_o} I^{0120}_{viwj}(0) - \frac{b_o^2 B_{16}}{a D_o} I^{1020}_{viwj}(0) - \frac{a^2 B_{22}}{b_o D_o} I^{0102}_{viwj}(0) \\ & - \frac{a B_{26}}{D_o}\left[I^{1002}_{viwj}(0) + 2 I^{0111}_{viwj}(0)\right] - \frac{2b_o B_{66}}{D_o} I^{1011}_{viwj}(0) \end{aligned} \tag{A1e}$$

$$\begin{aligned} k^{ww}_{ij} = {} & \frac{a^2 b_o^2 A_{22}}{\beta_o^2 D_o} I^{0000}_{wiwj}(2) + \frac{4a^2 b_o^2 A_{26}}{\beta_o R_{xy} D_o} I^{0000}_{wiwj}(1) \\ & + \frac{4a^2 b_o^2 A_{66}}{R_{xy}^2 D_o} I^{0000}_{wiwj}(0) - \frac{b_o^2 B_{12}}{\beta_o D_o}\left[I^{0020}_{wiwj}(1) + I^{2000}_{wiwj}(1)\right] \\ & - \frac{2b_o^2 B_{16}}{R_{xy} D_o}\left[I^{2000}_{wiwj}(0) + I^{0020}_{wiwj}(0)\right] \\ & - \frac{a^2 B_{22}}{\beta_o D_o}\left[I^{0002}_{wiwj}(1) + I^{0200}_{wiwj}(1)\right] \\ & - \frac{2ab_o B_{26}}{\beta_o D_o}\left[I^{0011}_{wiwj}(1) + I^{1100}_{wiwj}(1)\right] \\ & - \frac{2a^2 B_{26}}{R_{xy} D_o}\left[I^{0200}_{wiwj}(0) + I^{0002}_{wiwj}(0)\right] \\ & - \frac{4ab_o B_{66}}{R_{xy} D_o}\left[I^{1100}_{wiwj}(0) + I^{0011}_{wiwj}(0)\right] \\ & + \frac{b_o^2 D_{11}}{a^2 D_o} I^{2020}_{wiwj}(0) + \frac{D_{12}}{D_o}\left[I^{0220}_{wiwj}(0) + I^{2002}_{wiwj}(0)\right] \\ & + \frac{2b_o D_{16}}{a D_o}\left[I^{1120}_{wiwj}(0) + I^{2011}_{wiwj}(0)\right] + \frac{a^2 D_{22}}{b_o^2 D_o} I^{0202}_{wiwj}(0) \\ & + \frac{2a D_{26}}{b_o D_o}\left[I^{1102}_{wiwj}(0) + I^{0211}_{wiwj}(0)\right] + \frac{4 D_{66}}{D_o} I^{1111}_{wiwj}(0) \end{aligned} \tag{A1f}$$

$$m^{uu}_{ij} = I^{0000}_{uiuj}(0) \tag{A2a}$$

$$m^{vv}_{ij} = I^{0000}_{vivj}(0) \tag{A2b}$$

$$m^{ww}_{ij} = I^{0000}_{wiwj}(0) \tag{A2c}$$

where

$$I^{defg}_{uiuj}(\gamma) = \iint_{\bar{A}} \left[\frac{\beta_o}{R_y(\xi,\eta)}\right]^{\gamma} \frac{\partial^{d+e}\phi_i^u(\xi,\eta)}{\partial\xi^d \partial\eta^e} \frac{\partial^{f+g}\phi_j^u(\xi,\eta)}{\partial\xi^f \partial\eta^g}\, \mathrm{d}\xi\, \mathrm{d}\eta \tag{A3a}$$

$$I^{defg}_{uivj}(\gamma) = \iint_{\bar{A}} \left[\frac{\beta_o}{R_y(\xi,\eta)}\right]^{\gamma} \frac{\partial^{d+e}\phi_i^u(\xi,\eta)}{\partial\xi^d \partial\eta^e} \frac{\partial^{f+g}\phi_j^v(\xi,\eta)}{\partial\xi^f \partial\eta^g}\, \mathrm{d}\xi\, \mathrm{d}\eta \tag{A3b}$$

$$I^{defg}_{uiwj}(\gamma) = \iint_{\bar{A}} \left[\frac{\beta_o}{R_y(\xi,\eta)}\right]^{\gamma} \frac{\partial^{d+e}\phi_i^u(\xi,\eta)}{\partial\xi^d \partial\eta^e} \frac{\partial^{f+g}\phi_j^w(\xi,\eta)}{\partial\xi^f \partial\eta^g}\, \mathrm{d}\xi\, \mathrm{d}\eta \tag{A3c}$$

$$I^{defg}_{vivj}(\gamma) = \iint_{\bar{A}} \left[\frac{\beta_o}{R_y(\xi,\eta)}\right]^{\gamma} \frac{\partial^{d+e}\phi_i^v(\xi,\eta)}{\partial\xi^d \partial\eta^e} \frac{\partial^{f+g}\phi_j^v(\xi,\eta)}{\partial\xi^f \partial\eta^g}\, \mathrm{d}\xi\, \mathrm{d}\eta \tag{A3d}$$

$$I^{defg}_{viwj}(\gamma) = \iint_{\bar{A}} \left[\frac{\beta_o}{R_y(\xi,\eta)}\right]^{\gamma} \frac{\partial^{d+e}\phi_i^v(\xi,\eta)}{\partial\xi^d \partial\eta^e} \frac{\partial^{f+g}\phi_j^w(\xi,\eta)}{\partial\xi^f \partial\eta^g}\, \mathrm{d}\xi\, \mathrm{d}\eta \tag{A3e}$$

$$I^{defg}_{wiwj}(\gamma) = \iint_{\bar{A}} \left[\frac{\beta_o}{R_y(\xi,\eta)}\right]^{\gamma} \frac{\partial^{d+e}\phi_i^w(\xi,\eta)}{\partial\xi^d \partial\eta^e} \frac{\partial^{f+g}\phi_j^w(\xi,\eta)}{\partial\xi^f \partial\eta^g}\, \mathrm{d}\xi\, \mathrm{d}\eta \tag{A3f}$$

and $i, j = 1, 2, \ldots, m$, where m depends on the degree of polynomial. The reference plate flexural rigidity is $D_o = E_{11}h^3/12(1 - \nu_{12}\nu_{21})$, where E_{11} is the Young's modulus; and ν_{12} and ν_{21} are Poisson's ratios.[23] The integration domain $\bar{A}$ is the normalized planform area of the shallow conical shell in accordance with Eqs. (11a) and (11b).

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